THE SUM OF THE r'TH ROOTS OF FIRST n NATURAL NUMBERS AND NEW FORMULA FOR FACTORIAL

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Abstract. Using the simple properties of Riemman integrable functions, Ramanujan's formula for sum of the square roots of first n natural numbers has been generalized to include $r'th$ roots where r is any real number greater than 1. As an application we derive formula that gives factorial of positive integer n similar to Stirling's formula.

The formula for the sum of the square roots of first n natural numbers has been given by Srinivas Ramanujan ([Ra15]). Here we extend his result to the case of r'th roots, where r is a real number greater than 1.

Statement of result:

Theorem 0.1. *Let r be a real number with $r \geq 1$ and n be a positive integer. Then*

$$\sum_{x=1}^{x=n} x^{\frac{1}{r}} = \frac{r}{r+1}(n+1)^{\frac{1+r}{r}} - \frac{1}{2}(n+1)^{\frac{1}{r}} - \phi_n(r) \tag{1}$$

where ϕ_n is a function of r with n as a parameter. This function is bounded between 0 and $\frac{1}{2}$.

Proof. For a closed interval [a,b] we define partition of this interval as a set of points $x_0 = a, x_1, ..., x_{n-1} = b$ where $x_i < x_j$ whenever $i < j$. Now consider the closed interval [0,n] and consider a partition P of this interval,where P is a set $\{0, 1, 2, ..., n\}$.

Consider a function defined as $f(x)$=$x^{\frac{1}{r}}$.

We have,

$$I = \int_0^n f(x)dx = \lim_{\Delta x_i \to 0} \sum_{i=0}^{n-1} f(x_i)\Delta x_i$$

where

$$\Delta x_i = x_{i+1} - x_i$$

We define lower sum for partition P as:

$$L = \sum_{i=0}^{n-1} f(i)\Delta x_i = \sum_{i=0}^{n-1} i^{\frac{1}{r}}$$

Similarly,upper sum for P is

$$U = \sum_{i=1}^{n} f(i)\Delta x_i = \sum_{i=0}^{n-1} (i+1)^{\frac{1}{r}}$$

We write value of integral I as average of L and U with some correction term.

$$2I = L + U + \phi$$

$$\therefore 2\int_0^n x^{\frac{1}{r}} dx = \sum_{i=0}^{n-1} \left(i^{\frac{1}{r}} + (i+1)^{\frac{1}{r}}\right) + \phi$$

$$\therefore \frac{2r}{r+1}\left(x^{\frac{1+r}{r}}\right)_0^n = 0^{\frac{1}{r}} + 2\sum_{i=1}^{n-1} i^{\frac{1}{r}} + n^{\frac{1}{r}} + \phi$$

$$\therefore \sum_{i=1}^{n-1} i^{\frac{1}{r}} = \frac{r}{r+1} n^{\frac{1+r}{r}} - \frac{n^{\frac{1}{r}}}{2} - \phi$$

where the term of $\frac{1}{2}$ has been absorbed into ϕ.

$$\sum_{i=1}^{n} i^{\frac{1}{r}} = \frac{r}{r+1}(n+1)^{\frac{1+r}{r}} - \frac{1}{2}(n+1)^{\frac{1}{r}} - \phi \tag{2}$$

Taking limit of (2) as $r \to \infty$,$L.H.S. \to n$ and $R.H.S. \to (n + \frac{1}{2} - \phi)$, so that in the limit $\phi \to \frac{1}{2}$

On the other extreme, for $r = 1$,

$$\begin{aligned} R.H.S. &= \frac{1}{2}(n+1)^2 - \frac{1}{2}(n+1) - \phi \\ &= \frac{1}{2}(n^2 + n) - \phi \\ &= \frac{n(n+1)}{2} - \phi \end{aligned}$$

and,

$$L.H.S. = \frac{n(n+1)}{2}$$

This gives $\phi_n(1) = 0$

Since the difference between first and second term can easily be shown to be monotonic, we see that ϕ is bounded between between 0 and $\frac{1}{2}$ for $1 \leq r < \infty$ $\square$

As an application of the formula derived above, we derive formula to derive factorial of positive integer. We begin by taking derivative of (2) with respect to r. After rearranging the terms, we get,

(3)
$$\sum_{i=1}^{i=n} i^{\frac{1}{r}} \log i = \left[\frac{r}{r+1}(n+1) - \frac{1}{2}\right](n+1)^{\frac{1}{r}} \log(n+1) - \frac{r^2}{(r+1)^2}(n+1)^{\frac{1+r}{r}} + r^2\frac{d\phi}{dr}$$

After taking limit of this equation as $r \to \infty$, we get following equation:

$$\sum_{i=1}^{n} \log i = (n+\frac{1}{2})\log(n+1) - (n+1) + \lim_{r\to\infty} r^2\frac{d\phi}{dr} \tag{4}$$

In the above expression, $L.H.S$ is just $\log(n!)$. Let us assume that limit in the last term of the above equation exists and is finite and say that it is ξ. Then we can rewrite above equation as follows:

$$n! = (n+1)^{n+\frac{1}{2}} e^{-n-1} e^{\xi} \tag{5}$$

Numerically it turns out that the quantity e^{ξ} indeed converges to finite value, the value being close to $\sqrt{2\pi}$. This formula is similar to precise version of Stirling's formula ([St1]).

Equation (3) allows us to find one more interesting formula. After putting $r = 1$ in (3) and after little rearrangement, we get following beautiful formula:

$$\log\left[1^1.2^2...n^n\right] = \frac{n(n+1)}{2}\log(n+1) - \frac{1}{4}(n+1)^2 + \frac{d\phi}{dr}|_{r=1} \tag{6}$$

Numerically it turns out that quantity $\frac{d\phi}{dr}|_{r=1}$ is very small and can be neglected.

References

[Ra15] Ramanujan S., *On the sum of the square roots of the first n natural numbers.*, J. Indian Math. Soc., VII, (1915), 173-175.

[St1] Abramowitz, M. and Stegun, I. (2002), Handbook of Mathematical Functions.

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